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Exact multiplicity of solutions for discrete second order Neumann boundary value problems

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Abstract

Our concern is the second order difference equation $\Delta^2 u(t-1) + g(u(t)) = h(t)$ subject to the Neumann boundary conditions $\Delta u(0) = \Delta u(T) = 0$. Under convex/concave conditions imposed on g , some results on the exact numbers of solutions and positive solutions are established based on the discussions to the maximum and minimum numbers of (positive) solutions.

Keywords: difference equation; Neumann boundary value problem; exact numbers of solutions and positive solutions

1 Introduction

For $a, b \in \mathbb{Z}$ with $a < b$, define $[a, b]_{\mathbb{Z}} = \{a, a+1, a+2, \dots, b-1, b\}$. Consider the following Neumann boundary value problem:

$$\begin{cases} \Delta^2 u(t-1) + g(u(t)) = h(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \quad (1.1)$$

where $h : [1, T]_{\mathbb{Z}} \rightarrow \mathbb{R}$, $\Delta u(t) = u(t+1) - u(t)$ and $T > 3$ is a given positive integer. Our purpose is to find the exact number of solutions and positive solutions of (1.1).

In these last years, the existence and multiplicity of solutions for nonlinear discrete problems subject to various boundary value conditions have been widely studied by using different abstract methods such as critical point theory, fixed point theorems, lower and upper solutions method, and Brower degree (see, e.g., [1–17] and the references therein). All these results are about the unique solution, or the minimum amount of solutions, and positive solutions. To the best of our knowledge, there is no report on the exact number of solutions for discrete boundary value problems.

For BVPs of differential equations, there are many papers concerned with the bifurcation values and exact multiplicities of solutions and positive solutions by bifurcation theory, quadrature method, time-map analysis and otherwise. See [18–29] and the references therein. For difference equations, however, the loss of continuity puts some methods used well in differential equations, such as the quadrature method and its time-map analysis, out of action. Therefore, it is very meaningful to study the exact number of solutions for

discrete boundary value problems. In this paper, based on the discussions to the maximum and minimum numbers of (positive) solutions, we establish some results on the exact number of (positive) solutions of (1.1) under convex/concave conditions.

The remaining part of this paper is organized as follows. In Section 2, under the Neumann boundary conditions $\Delta u(0) = \Delta u(T) = 0$, we show the Green's functions of the linear difference operators $-\Delta^2 u(t-1) + Lu(t)$ and $\Delta^2 u(t-1) + Ku(t)$, where L and K are two positive constants with $K < 4 \sin^2 \frac{\pi}{2T}$. Then, in Section 3, we make some estimates on the maximum number of solutions of (1.1), and, in Section 4, we establish some results on the minimum numbers of solutions and positive solutions of (1.1). Finally, we give the results on the exact multiplicities of solutions and positive solutions of (1.1) in Section 5.

2 Green's functions

In this section, we show the Green's functions of the linear difference operators $-\Delta^2 u(t-1) + Lu(t)$ and $\Delta^2 u(t-1) + Ku(t)$ satisfying the Neumann boundary conditions $\Delta u(0) = \Delta u(T) = 0$.

Let $0 < K < 4 \sin^2 \frac{\pi}{2T}$. Consider the Neumann boundary value problem

$$\begin{cases} \Delta^2 u(t-1) + Ku(t) = h(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \quad (2.1)$$

It is easy to see that the corresponding homogeneous equation $\Delta^2 u(t-1) + Ku(t) = 0$ has two independent real solutions $\cos \theta t$ and $\sin \theta t$, where θ is defined by

$$\theta = \arccos\left(1 - \frac{K}{2}\right), \quad 0 < \theta < \frac{\pi}{T}. \quad (2.2)$$

Lemma 2.1 *Problem (2.1) has a unique solution*

$$u(t) = \sum_{s=1}^T G_1(t, s)h(s), \quad (2.3)$$

where

$$G_1(t, s) = \frac{1}{\sin \theta \sin \theta T} \begin{cases} \cos \frac{2s-1}{2}\theta \cos \frac{2T+1-2t}{2}\theta, & 1 \leq s \leq t \leq T+1, \\ \cos \frac{2t-1}{2}\theta \cos \frac{2T+1-2s}{2}\theta, & 0 \leq t \leq s \leq T. \end{cases}$$

In addition, if $0 < K < 4 \sin^2 \frac{\pi}{4T}$, then $G_1(t, s) > 0$ for all $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$.

Proof First, we show that u given by (2.3) solves (2.1). By $G_1(0, s) = G_1(1, s)$ we know that $u(0) = u(1)$, i.e., $\Delta u(0) = 0$. Similarly, $G_1(T, s) = G_1(T+1, s)$ implies that $\Delta u(T) = 0$. Now we show that u satisfies the equation $\Delta^2 u(t-1) + Ku(t) = h(t)$, i.e., $u(t+1) + (K-2)u(t) + u(t-1) = h(t)$, $t \in [1, T]_{\mathbb{Z}}$. Since

$$\begin{aligned} & u(t+1) + (K-2)u(t) + u(t-1) \\ &= \sum_{s=1}^T [G_1(t+1, s) + (K-2)G_1(t, s) + G_1(t-1, s)]h(s) \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{s=1}^{t-1} + \sum_{s=t+1}^T \right) [G_1(t+1, s) + (K-2)G_1(t, s) + G_1(t-1, s)]h(s) \\
&\quad + [G_1(t+1, t) + (K-2)G_1(t, t) + G_1(t-1, t)]h(t) \\
&:= \sum_{s=1}^{t-1} I_1(t, s)h(s) + \sum_{s=t+1}^T I_2(t, s)h(s) + I_3(t)h(t),
\end{aligned}$$

we just need to verify that $I_1 \equiv 0$, $I_2 \equiv 0$, and $I_3 \equiv 1$. In fact, by the fact that $\cos \theta t$ solves the homogeneous equation $\Delta^2 u(t-1) + Ku(t) = 0$, we know that, for all $t \in [1, T]_{\mathbb{Z}}$ and $1 \leq s \leq t-1$,

$$\begin{aligned}
&I_1(t, s) \sin \theta \sin \theta T \\
&= \cos \frac{(2s-1)\theta}{2} \left[\cos \frac{(2T-2t-1)\theta}{2} + (K-2) \cos \frac{(2T+1-2t)\theta}{2} \right. \\
&\quad \left. + \cos \frac{(2T+3-2t)\theta}{2} \right] \\
&= 0.
\end{aligned}$$

This implies $I_1 \equiv 0$. Similarly $I_2 \equiv 0$. Finally,

$$\begin{aligned}
&I_3(t) \sin \theta \sin \theta T \\
&= \cos \frac{(2(t-1)-1)\theta}{2} \cos \frac{(2T+1-2t)\theta}{2} - \cos \frac{(2t-1)\theta}{2} \cos \frac{(2T+1-2(t-1))\theta}{2} \\
&= \frac{1}{2} [\cos(T-1)\theta - \cos(T+1)\theta] \\
&= \sin \theta \sin \theta T
\end{aligned}$$

holds for all $t \in [1, T]_{\mathbb{Z}}$, which implies $I_3 \equiv 1$.

Since the corresponding homogeneous problem has only the trivial zero solution, problem (2.1) has a unique solution u , which is given by (2.1). Clearly, if $0 < K < 4 \sin^2 \frac{\pi}{4T}$, then $0 < \theta < \frac{\pi}{2T}$ and hence $G_1(t, s) > 0$ for all $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$. The proof is complete. \square

Let $L > 0$. Consider the Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) + Lu(t) = h(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \quad (2.4)$$

For convenience, let

$$A = \frac{1}{2}(L + 2 + \sqrt{L^2 + 4L}), \quad \rho = (A^T - A^{-T})(A^2 - 1).$$

Lemma 2.2 *Problem (2.4) has a unique solution*

$$u(t) = \sum_{s=1}^T G_2(t, s)h(s), \quad (2.5)$$

where

$$G_2(t, s) = \frac{1}{\rho} \begin{cases} (A^s + A^{-s+1})(A^{t-T} + A^{T-t+1}), & 1 \leq s \leq t \leq T+1, \\ (A^t + A^{-t+1})(A^{s-T} + A^{T-s+1}), & 0 \leq t \leq s \leq T, \end{cases}$$

and $G_2(t, s) > 0$ for all $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$.

Proof First, we show that u given by (2.5) solves (2.4). By $G_2(0, s) = G_2(1, s)$ we know $\Delta u(0) = 0$ and by $G_2(T, s) = G_2(T+1, s)$, $\Delta u(T) = 0$. Now we show that u satisfies the equation $-\Delta^2 u(t-1) + Lu(t) = h(t)$, i.e., $u(t+1) - (L+2)u(t) + u(t-1) = -h(t)$, $t \in [1, T]_{\mathbb{Z}}$. Since

$$\begin{aligned} & u(t+1) - (L+2)u(t) + u(t-1) \\ &= \sum_{s=1}^T [G_2(t+1, s) - (L+2)G_2(t, s) + G_2(t-1, s)]h(s) \\ &= \left(\sum_{s=1}^{t-1} + \sum_{s=t+1}^T \right) [G_2(t+1, s) - (L+2)G_2(t, s) + G_2(t-1, s)]h(s) \\ &\quad + [G_2(t+1, t) - (L+2)G_2(t, t) + G_2(t-1, t)]h(t) \\ &:= \sum_{s=1}^{t-1} I_1(t, s)h(s) + \sum_{s=t+1}^T I_2(t, s)h(s) + I_3(t)h(t), \end{aligned}$$

we just need to verify that $I_1 \equiv 0$, $I_2 \equiv 0$, and $I_3 \equiv -1$. In fact, by the fact that A is a real root of the equation $\ell(\lambda) := \lambda^2 - (L+2)\lambda + 1 = 0$, we know that, for all $t \in [1, T]_{\mathbb{Z}}$ and $1 \leq s \leq t-1$,

$$\rho I_1(t, s) = (A^s + A^{-s+1})(A^{t-T-1} + A^{T-t})\ell(A) = 0,$$

holds, which implies $I_1 \equiv 0$. Similarly $I_2 \equiv 0$. For I_3 , we have, for all $t \in [1, T]_{\mathbb{Z}}$,

$$\begin{aligned} \rho I_3(t) &= (A^t + A^{-t+1})[A^{t-T-1}(A^2 - (L+2)A) + A^{T-t}(1 - (L+2)A)] \\ &\quad + (A^{t-1} + A^{-t+2})(A^{t-T} + A^{T-t+1}) \\ &= (A^t + A^{-t+1})(-A^{t-T-1} - A^{T-t+2}) + (A^{t-1} + A^{-t+2})(A^{t-T} + A^{T-t+1}) \\ &= -(A^2 - 1)(A^T - A^{-T}) \\ &= -\rho, \quad t \in [1, T]_{\mathbb{Z}}. \end{aligned}$$

Thus, $I_3 \equiv -1$.

It is easy to check that the corresponding homogeneous problem has only the trivial zero solution. So problem (2.4) has a unique solution u which is given by (2.5). Finally, since $A > 1$, we have, for all $(t, s) \in [0, T+1]_{\mathbb{Z}} \times [1, T]_{\mathbb{Z}}$, $G_2(t, s) > 0$. The proof is complete. \square

3 Estimates on the maximum number of solutions

In this section, we make estimations on the maximum number of solutions of problem (1.1). First, we prove some lemmas for later use. The first lemma is the discrete Sobolev inequality.

Lemma 3.1 Assume $u : [0, T + 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ satisfies $\Delta u(T) = 0$ and there exists $t_0 \in [2, T - 1]$ such that $u(t_0) = 0$. Then

$$\sum_{t=t_0+1}^T |u(t)|^q \leq \left(\sum_{t=t_0+1}^T (t - t_0)^{\frac{q}{p}} \right) \left(\sum_{t=t_0+1}^T |\Delta u(t-1)|^q \right),$$

$$\sum_{t=1}^{t_0} |u(t)|^q \leq \left(\sum_{t=1}^{t_0} (t_0 - t)^{\frac{q}{p}} \right) \left(\sum_{t=1}^{t_0} |\Delta u(t-1)|^q \right),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof For $t > t_0$, since $u(t) = \sum_{\tau=t_0}^{t-1} \Delta u(\tau) + u(t_0) = \sum_{\tau=t_0}^{t-1} \Delta u(\tau)$, we have by the Hölder inequality

$$|u(t)| \leq \sum_{\tau=t_0}^{t-1} |\Delta u(\tau)| \leq (t - t_0)^{\frac{1}{p}} \left(\sum_{\tau=t_0}^{t-1} |\Delta u(\tau)|^q \right)^{\frac{1}{q}},$$

which implies that

$$\sum_{t=t_0+1}^T |u(t)|^q \leq \left(\sum_{t=t_0+1}^T (t - t_0)^{\frac{q}{p}} \right) \left(\sum_{\tau=t_0+1}^T |\Delta u(\tau-1)|^q \right)$$

by the boundary condition $\Delta u(T) = 0$. On the other hand, for $t < t_0$, since $u(t) = u(t_0) - \sum_{\tau=t}^{t_0-1} \Delta u(\tau) = -\sum_{\tau=t}^{t_0-1} \Delta u(\tau)$, we have

$$|u(t)| \leq \sum_{\tau=t}^{t_0-1} |\Delta u(\tau)| \leq (t_0 - t)^{\frac{1}{p}} \left(\sum_{\tau=t}^{t_0-1} |\Delta u(\tau)|^q \right)^{\frac{1}{q}},$$

which implies that

$$\sum_{t=1}^{t_0} |u(t)|^q \leq \left(\sum_{t=1}^{t_0} (t_0 - t)^{\frac{q}{p}} \right) \left(\sum_{t=1}^{t_0} |\Delta u(t-1)|^q \right).$$

The proof is complete. \square

Definition 3.1 We say that $u : [0, T + 1]_{\mathbb{Z}} \rightarrow \mathbb{R}$ has a generalized zero at $t_0 \in [0, T + 1]_{\mathbb{Z}}$ provided that $u(t_0) = 0$ if $t_0 = 0$ and if $t_0 \in [1, T + 1]_{\mathbb{Z}}$ either $u(t_0) = 0$ or $u(t_0 - 1)u(t_0) < 0$.

Lemma 3.2 Let $p : [1, T]_{\mathbb{Z}} \rightarrow \mathbb{R}$ and $p(t) \neq 0$ on $[1, T]_{\mathbb{Z}}$. Consider the following problem:

$$\begin{cases} \Delta^2 u(t-1) + p(t)u(t) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \quad (3.1)$$

If $p(t) < \frac{4}{T^2 - 1}$ for $t \in [1, T]_{\mathbb{Z}}$, then every nontrivial solution $u(t)$ of (3.1) has no generalized zero on $[0, T + 1]_{\mathbb{Z}}$.

Proof Let u be a nontrivial solution $u(t)$ of (3.1). By the boundary conditions $\Delta u(0) = \Delta u(T) = 0$, we know that $u(0) = u(1) \neq 0$ and $u(T) = u(T+1) \neq 0$. Otherwise, $u \equiv 0$ on $[0, T+1]_{\mathbb{Z}}$. Suppose that there exists $t_0 \in [2, T-1]_{\mathbb{Z}}$ such that $u(t_0) = 0$. Multiplying the equation in (3.1) with $u(t)$ and summing both sides from 1 to t_0 , we have

$$\sum_{t=1}^{t_0} [\Delta u(t-1)]^2 = \sum_{t=1}^{t_0} p(t)u^2(t) \leq \max_{1 \leq t \leq t_0} p_+(t) \sum_{t=1}^{t_0} u^2(t),$$

where p_+ is the positive part of p . Similarly, summing both sides from t_0+1 to T , we have

$$\sum_{t=t_0+1}^T [\Delta u(t-1)]^2 = \sum_{t=t_0+1}^T p(t)u^2(t) \leq \max_{t_0+1 \leq t \leq T} p_+(t) \sum_{t=t_0+1}^T u^2(t).$$

Thus, by Lemma 3.1, we have

$$\max_{1 \leq t \leq t_0} p_+(t) \left(\sum_{t=1}^{t_0} (t_0 - t) \right) \geq 1, \quad \max_{t_0+1 \leq t \leq T} p_+(t) \left(\sum_{t=t_0+1}^T (t - t_0) \right) \geq 1.$$

It follows that

$$\max_{1 \leq t \leq T} p_+(t) \geq \max \left\{ \left(\sum_{t=1}^{t_0} (t_0 - t) \right)^{-1}, \left(\sum_{t=t_0+1}^T (t - t_0) \right)^{-1} \right\} \geq \frac{4}{T^2 - 1},$$

which is a contradiction with the assumption $p(t) < \frac{4}{T^2-1}$ for $t \in [1, T]_{\mathbb{Z}}$.

Note that the equation $\Delta^2 u(t-1) + p(t)u(t) = 0$ is disconjugate on $\in [0, T+1]_{\mathbb{Z}}$ since $p(t) < \frac{4}{T^2-1}$ for $t \in [1, T]_{\mathbb{Z}}$ (see Corollary 6.9 in [30]). It implies that the nontrivial u has at most one generalized zero on $\in [0, T+1]_{\mathbb{Z}}$. Set $z(t) = \frac{u(t+1)}{u(t)}$. Then $z(t)$ satisfies

$$z(t) + \frac{1}{z(t-1)} = 2 - p(t), \quad z(0) = z(T) = 1.$$

If there exists $t_0 \in [1, T]_{\mathbb{Z}}$ such that $z(t_0) < 0$, then the boundary conditions $z(0) = z(T) = 1$ implies that $z(t)$ has at least two generalized zeros. Thus, $u(t)$ has at least two generalized zeros, which contradicts with that $u(t)$ is disconjugate. The proof is complete. \square

Lemma 3.3 Let $p_1, p_2 : [1, T]_{\mathbb{Z}} \rightarrow \mathbb{R}$ satisfy the conditions of Lemma 3.2. Assume $p_1(t) \leq p_2(t)$ on $[1, T]_{\mathbb{Z}}$ and there exists $t_0 \in [1, T]_{\mathbb{Z}}$ such that $p_1(t_0) < p_2(t_0)$. Then at least one of the problems

$$\begin{cases} \Delta^2 u(t-1) + p_i(t)u(t) = 0, & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \quad i = 1, 2, \quad (3.2)$$

has only the trivial solution.

Proof By way of contradiction, assume that u_1 and u_2 are the nontrivial solutions of problems (3.2) corresponding to p_1 and p_2 , respectively. By Lemma 3.2, we know that $u_1(t)$ and

$u_2(t)$ both have no generalized zero on $t \in [0, T+1]_{\mathbb{Z}}$. It follows by $p_1(t) - p_2(t) \leq 0$ and $p_1(t) - p_2(t) \neq 0, t \in [1, T]_{\mathbb{Z}}$, that

$$\sum_{t=1}^T [p_1(t) - p_2(t)] u_1(t) u_2(t) \neq 0. \quad (3.3)$$

On the other hand, by

$$u_2(t) [\Delta^2 u_1(t-1) + p_1(t) u_1(t)] = 0,$$

$$u_1(t) [\Delta^2 u_2(t-1) + p_2(t) u_2(t)] = 0,$$

we have

$$[u_2(t) \Delta^2 u_1(t-1) - u_1(t) \Delta^2 u_2(t-1)] + [p_1(t) - p_2(t)] u_1(t) u_2(t) = 0.$$

Summing both sides from 1 to T , we get

$$\sum_{t=1}^T [p_1(t) - p_2(t)] u_1(t) u_2(t) = 0,$$

which is a contradiction with (3.3). The proof is complete. \square

Lemma 3.4 Assume $g \in C^1(\mathbb{R})$ and $g'(x) < \frac{4}{T^2-1}$ for all $x \in \mathbb{R}$. Let u_1 and u_2 are two distinct solutions of (1.1), then $u_1(t) - u_2(t)$ has no generalized zero on $[0, T+1]_{\mathbb{Z}}$ and hence u_1 and u_2 are strictly ordered.

Proof Set $v = u_1 - u_2$. Then v is a nontrivial solution of (3.1) with

$$p(t) = \begin{cases} \frac{g(u_1(t)) - g(u_2(t))}{u_1(t) - u_2(t)}, & u_1(t) \neq u_2(t), \\ g'(u_1(t)), & u_1(t) = u_2(t). \end{cases}$$

Since $g'(x) < \frac{4}{T^2-1}$, we know $p(t) < \frac{4}{T^2-1}$ for all $t \in [1, T]_{\mathbb{Z}}$. Thus, Lemma 3.2 implies that $v(t)$ has no generalized zero on $[0, T+1]_{\mathbb{Z}}$. Therefore, u_1 and u_2 are strictly ordered. The proof is complete. \square

Now, we show our main results of this section. First, if g is strictly convex on \mathbb{R} , we have the following result.

Theorem 3.5 Assume $g \in C^1(\mathbb{R})$, $g'(x) < \frac{4}{T^2-1}$ for all $x \in \mathbb{R}$ and g' is strictly increasing. Then problem (1.1) has at most two solutions.

Proof Suppose that problem (1.1) has three solutions u_1 , u_2 , and u_3 . By Lemma 3.4, we assume $u_1(t) < u_2(t) < u_3(t)$, $t \in [0, T+1]_{\mathbb{Z}}$. Let $v_1 = u_2 - u_1$, $v_2 = u_3 - u_2$, then v_1 and v_2 solve problem (3.1) with, respectively,

$$p(t) = p_1(t) := \frac{g(u_2(t)) - g(u_1(t))}{u_2(t) - u_1(t)}$$

and

$$p(t) = p_2(t) := \frac{g(u_3(t)) - g(u_2(t))}{u_3(t) - u_2(t)}.$$

Since g' is strictly increasing, we have $p_1(t) < p_2(t) < \frac{4}{T^2-1}$. By Lemma 3.3, either $v_1(t) \equiv 0$ or $v_2(t) \equiv 0$, which is a contradiction. The proof is complete. \square

Similarly, if g is strictly concave on \mathbf{R} , we have the following result.

Theorem 3.6 Assume $g \in C^1(\mathbf{R})$, $g'(x) < \frac{4}{T^2-1}$ for all $x \in \mathbf{R}$ and g' is strictly decreasing. Then problem (1.1) has at most two solutions.

4 On the minimum number of solutions and positive solutions

In this section, we consider the minimum numbers of solution and positive solutions of (1.1). First, we consider the nonexistence, existence and multiplicity of solutions of (1.1) when $g \in C^1(\mathbf{R})$ and g has a unique global minimum value on \mathbf{R} .

Theorem 4.1 Let $g \in C^1(\mathbf{R})$, $\lim_{x \rightarrow -\infty} g(x) = a$, $\lim_{x \rightarrow +\infty} g(x) = b$, and $\lambda_* := g(x_0) = \min_{x \in \mathbf{R}} g(x)$. Here, a, b may be $+\infty$. Assume $g(x) > \lambda_*$ for all $x \in \mathbf{R} \setminus \{x_0\}$ and $\lambda_* < \min\{a, b\}$.

(1) If one of the following conditions is satisfied, then problem (1.1) has no solution:

(1-i) $h(t) \leq \lambda_*$ and $h(t) \neq \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$;

(1-ii) $\max\{a, b\} < +\infty$, $h(t) \geq \max\{a, b\}$ for all $t \in [1, T]_{\mathbf{Z}}$ and $\max\{a, b\} > g(x)$ for all $x \in \mathbf{R}$.

(2) If $h(t) \equiv \lambda_*$ on $[1, T]_{\mathbf{Z}}$, then problem (1.1) has exactly one solution.

(3) If $\lambda_* \leq h(t) < \min\{a, b\}$, $h(t) \neq \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$, then problem (1.1) has at least two solutions.

(4) If one of the following conditions is satisfied, then problem (1.1) has at least one solution:

(4-i) $a \neq b$, $\min\{a, b\} < +\infty$ and $h(t) \equiv \min\{a, b\}$, $t \in [1, T]_{\mathbf{Z}}$;

(4-ii) $b < h(t) < a$, $t \in [1, T]_{\mathbf{Z}}$;

(4-iii) $a < h(t) < b$, $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$.

Proof (1-i) Since $h(t) \leq \lambda_*$ and $h(t) \neq \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$, we have $\sum_{t=1}^T h(t) < T\lambda_*$. Assume problem (1.1) has a solution u . Since $g(u(t)) \geq \lambda_*$, $t \in [1, T]_{\mathbf{Z}}$, $\sum_{t=1}^T g(u(t)) \geq T\lambda_*$. Summing both sides of the equation $\Delta^2 u(t-1) + g(u(t)) = h(t)$ from 1 to T , we have by the boundary conditions $\Delta u(0) = \Delta u(T) = 0$,

$$T\lambda_* \leq \sum_{t=1}^T g(u(t)) = \sum_{t=1}^T h(t) < T\lambda_*,$$

which is a contradiction.

(1-ii) Suppose (1.1) has a solution u . Then $g(u(t)) < \max\{a, b\}$ for all $t \in [1, T]_{\mathbf{Z}}$, which implies that $\Delta^2 u(t-1) = h(t) - g(u(t)) > 0$ for $t \in [1, T]_{\mathbf{Z}}$. Thus, we have the contradiction that $\Delta u(T) > \Delta u(0)$.

(2) It is easy to see that $u(t) \equiv x_0$ is a solution of (1.1). Assume that v is also a solution of (1.1). Since $g(v(t)) \geq \lambda_*$, $t \in [1, T]_{\mathbf{Z}}$, we have $\Delta^2 v(t-1) \leq 0$, $t \in [1, T]_{\mathbf{Z}}$. It follows by the

boundary conditions $\Delta v(0) = \Delta v(T) = 0$ that $\Delta v(t) \equiv 0$, $t \in [0, T]_{\mathbb{Z}}$. Thus, $g(v(t)) \equiv g(x_0)$. Therefore, $v(t) \equiv x_0$, $t \in [0, T+1]_{\mathbb{Z}}$, since $g(x) > g(x_0)$ for all $x \in \mathbb{R} \setminus \{x_0\}$.

(3) Without loss of generality, we assume $b \leq a$. Since $\lambda_* \leq h(t) < b$, $h(t) \neq \lambda_*$, $t \in [1, T]_{\mathbb{Z}}$, we see by $\lim_{x \rightarrow -\infty} g(x) = a$ and $\lim_{x \rightarrow +\infty} g(x) = b$ that there exist $c_1 < x_0 < c_2$ such that

$$g(c_1) > h(t), \quad g(c_2) > h(t), \quad t \in [1, T]_{\mathbb{Z}}. \quad (4.1)$$

Set $u_1(t) \equiv c_1$, $u_2(t) \equiv x_0$, $u_3(t) \equiv c_2$, $t \in [0, T+1]_{\mathbb{Z}}$. We prove that problem (1.1) has at least two solutions u_1^* , u_2^* : $u_1^* \in [u_1, u_2]$, $u_2^* \in [u_2, u_3]$. Here, $[u_1, u_2]$ and $[u_2, u_3]$ denote order intervals.

First, we show that $u_2(t)$ is not a solution of (1.1). In fact, if it is not true, then summing both sides of the equation of (1.1) yields a contradiction:

$$T\lambda_* = \sum_{t=1}^T g(u_2(t)) = \sum_{t=1}^T h(t) > T\lambda_*.$$

Second, we prove that (1.1) has at least one solution $u_1^* \in [u_1, u_2]$. Let $E = \{u : [0, T+1]_{\mathbb{Z}} \rightarrow \mathbb{R}\}$ with the norm $\|u\| = \max_{t \in [0, T+1]_{\mathbb{Z}}} |u(t)|$. Since g' is continuous on the bounded closed interval $[c_1, x_0]$, there exists $L > 0$ such that $g'(x) > -L$ for $x \in [c_1, x_0]$. Consider the following problem:

$$\begin{cases} -\Delta^2 u(t-1) + Lu(t) = g(u(t)) + Lu(t) - h(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \quad (4.2)$$

By Lemma 2.2, (4.2) is equivalent to $u(t) = Su(t)$. Here, $S : E \rightarrow E$ is defined by

$$Su(t) = \sum_{s=1}^T G_2(t, s) [g(u(s)) + Lu(s) - h(s)], \quad t \in [0, T+1]_{\mathbb{Z}}. \quad (4.3)$$

It is easy to see that S is continuous. Since $u_1(t)$ satisfies

$$\begin{cases} -\Delta^2 u(t-1) + Lu(t) = Lu_1(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases}$$

we see by (4.1) and the positivity of $G_2(t, s)$ that

$$\begin{aligned} Su_1(t) &= \sum_{s=1}^T G_2(t, s) [g(u_1(s)) + Lu_1(s) - h(s)] \\ &> \sum_{s=1}^T G_2(t, s) Lu_1(s) = u_1(t), \quad t \in [0, T+1]_{\mathbb{Z}}, \end{aligned}$$

which shows that u_1 is a lower solution of the operator S . Similarly, one can check that $Su_2(t) \leq u_2(t)$ and hence u_2 is an upper solution of S . On the other hand, S is an increasing operator defined on $[u_1, u_2]$. In fact, for $v_1, v_2 \in [u_1, u_2]$ with $v_1 \leq v_2$, we have

$$\begin{aligned} Sv_2(t) - Sv_1(t) &= \sum_{s=1}^T G_2(t, s) [g(v_2(s)) - g(v_1(s)) + L(v_2(s) - v_1(s))] \\ &\geq 0, \quad t \in [0, T+1]_{\mathbb{Z}}, \end{aligned}$$

by $g'(x) > -L$ for $x \in [c_1, x_0]$. Therefore, S has at least one solution $u_1^* \in [u_1, u_2]$ by the fixed point theorem of increasing operator in ordered Banach spaces due to Amann [31]. So, (4.2), and hence (1.1), has at least one solution $u_1^* \in [u_1, u_2]$.

Now, we prove that (1.1) has at least one solution $u_2^* \in [u_2, u_3]$. Since $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for $x \in [x_0, +\infty)$, there exists $K > 0$ such that

$$g'(x) < K < 4 \sin^2 \frac{\pi}{4T}, \quad x \in [x_0, c_2]. \quad (4.4)$$

Consider the following problem:

$$\begin{cases} \Delta^2 u(t-1) + Ku(t) = h(t) - g(u(t)) + Ku(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0. \end{cases} \quad (4.5)$$

By Lemma 2.1, $G_1(t, s) > 0$, and (4.5) is equivalent to $u(t) = Qu(t)$. Here, $Q: E \rightarrow E$ is defined by

$$Qu(t) = \sum_{s=1}^T G_1(t, s) [h(s) - g(u(s)) + Ku(s)], \quad t \in [0, T+1]_{\mathbb{Z}}. \quad (4.6)$$

Similar to the discussion of the operator S , Q has at least one solution $u_2^* \in [u_2, u_3]$, which is a solution of (1.1). Therefore, problem (1.1) has at least two solutions.

(4-i) Without loss of generality, we assume $b < a$. Since $h(t) \equiv b$, $t \in [1, T]_{\mathbb{Z}}$, there exists $c_1 < x_0$ such that $g(c_1) = b$. Obviously, $u(t) \equiv c_1$ is a solution of (1.1).

(4-ii) Since $b < h(t) < a$, $t \in [1, T]_{\mathbb{Z}}$, there exist $c_1 < c_2 < x_0$ such that $g(c_1) > h(t)$, $g(c_2) < h(t)$, $t \in [1, T]_{\mathbb{Z}}$. By the continuity of g' , there exists $L > 0$ such that $g'(x) > -L$, $x \in [c_1, c_2]$. Let $u_1(t) \equiv c_1$, $u_2(t) \equiv c_2$. Considering the operator S defined as (4.3) on $[u_1, u_2]$, one can see that S has a fixed point $u^* \in [u_1, u_2]$, which is a solution of (1.1).

(4-iii) Since $a < h(t) < b$, $t \in [1, T]_{\mathbb{Z}}$, there exist $x_0 < c_1 < c_2$ such that $g(c_1) < h(t)$, $g(c_2) > h(t)$, $t \in [1, T]_{\mathbb{Z}}$. By $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for $x \geq x_0$, there exists $K > 0$ such that $g'(x) < K < 4 \sin^2 \frac{\pi}{4T}$, $x \in [c_1, c_2]$. Let $u_1(t) \equiv c_1$, $u_2(t) \equiv c_2$. Considering the operator Q defined as (4.5) on $[u_1, u_2]$, one can see that Q has a fixed point $u^* \in [u_1, u_2]$, which is a solution of (1.1).

The proof is complete. \square

Similarly, if $g \in C^1(\mathbb{R})$ and g has a unique global maximum value on \mathbb{R} , we have the following result.

Theorem 4.2 Let $g \in C^1(\mathbb{R})$, $\lim_{x \rightarrow -\infty} g(x) = a$, $\lim_{x \rightarrow +\infty} g(x) = b$, and $\lambda^* := g(x_0) = \max_{x \in \mathbb{R}} g(x)$. Here, a, b may be $-\infty$. Assume $g(x) < \lambda^*$ for all $x \in \mathbb{R} \setminus \{x_0\}$ and $\lambda^* > \max\{a, b\}$.

(1) If one of the following conditions is satisfied, then problem (1.1) has no solution:

(1-i) $h(t) \geq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$;

(1-ii) $\min\{a, b\} > -\infty$, $h(t) \leq \min\{a, b\}$ for $t \in [1, T]_{\mathbb{Z}}$ and $\min\{a, b\} < g(x)$ for $x \in \mathbb{R}$.

(2) If $h(t) \equiv \lambda^*$ on $[1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one solution.

(3) If $\max\{a, b\} < h(t) \leq \lambda^*$, $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$; and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $(-\infty, x_0]$, then problem (1.1) has at least two solutions.

(4) If one of the following conditions is satisfied, then problem (1.1) has at least one solution:

- (4-i) $a \neq b$, $\max\{a, b\} > -\infty$ and $h(t) \equiv \max\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$;
- (4-ii) $a < h(t) < b$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $(-\infty, x_0]$;
- (4-iii) $b < h(t) < a$ for $t \in [1, T]_{\mathbf{Z}}$.

Now, we consider the positive solutions of (1.1). Let $\langle 0, +\infty \rangle$ denote $(0, +\infty)$ or $[0, \infty)$. First, if $g \in C^1(\langle 0, +\infty \rangle, \mathbf{R})$ and g has a unique global minimum value on $\langle 0, +\infty \rangle$, we have the following result.

Theorem 4.3 *Let $g \in C^1(\langle 0, +\infty \rangle, \mathbf{R})$, $\lim_{x \rightarrow 0^+} g(x) = a$, $\lim_{x \rightarrow +\infty} g(x) = b$. Here, a, b may be $+\infty$. Assume that there exists $x_0 > 0$ such that*

$$\lambda_* := g(x_0) = \min_{x \in (0, +\infty)} g(x),$$

and $g(x) > \lambda_*$ for all $x \in \langle 0, +\infty \rangle \setminus \{x_0\}$, $\lambda_* < \min\{a, b\}$.

(I) Suppose $\lambda_* \geq 0$.

(1) If one of the following conditions is satisfied, then problem (1.1) has no positive solution:

- (1-i) $h(t) \leq \lambda_*$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$;
- (1-ii) $\max\{a, b\} < +\infty$, $h(t) \geq \max\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$, and $g(x) < \max\{a, b\}$ for all $x \in (0, +\infty)$.

(2) If $h(t) \equiv \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$, then problem (1.1) has exactly one positive solution.

(3) If $\lambda_* \leq h(t) < \min\{a, b\}$, $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, +\infty)$, then problem (1.1) has at least two positive solutions.

(4) If one of the following conditions is satisfied, then problem (1.1) has at least one positive solution:

- (4-i) $a \neq b$, $\min\{a, b\} < +\infty$, $h(t) \equiv \min\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$;
- (4-ii) $b < h(t) < a$ for $t \in [1, T]_{\mathbf{Z}}$;
- (4-iii) $a < h(t) < b$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$.

(II) Suppose $\lambda_* < 0$.

(5) If one of the following conditions is satisfied, then problem (1.1) has no positive solution:

- (5-i) $h(t) \leq \lambda_*$ for $t \in [1, T]_{\mathbf{Z}}$;
- (5-ii) $\max\{a, b\} < +\infty$ and $h(t) \geq \max\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$, and $g(x) < \max\{a, b\}$ for all $x \in (0, +\infty)$.

(6) If one of the following conditions is satisfied, then problem (1.1) has at least two positive solutions:

- (6-i) $a > 0$, $b > 0$, $0 < h(t) < \min\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$;
- (6-ii) $a > 0$, $b > 0$, and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbf{Z}}$.

(7) If one of the following conditions is satisfied, then problem (1.1) has at least one positive solution:

- (7-i) $0 < \min\{a, b\} < +\infty$, $a \neq b$, and $h(t) \equiv \min\{a, b\}$ for $t \in [1, T]_{\mathbf{Z}}$;
- (7-ii) $0 < b < a$ and $b < h(t) < a$ for $t \in [1, T]_{\mathbf{Z}}$;
- (7-iii) $0 < a < b$, $a < h(t) < b$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$;
- (7-iv) $a \leq 0$, $b > 0$, and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbf{Z}}$;
- (7-v) $a \leq 0$, $b > 0$, $0 < h(t) < b$ for $t \in [1, T]_{\mathbf{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $[x_0, \infty)$;
- (7-vi) $a > 0$, $b < 0$, and $0 < h(t) < a$ for $t \in [1, T]_{\mathbf{Z}}$.

Proof We only show the proofs of (3), (6), and (7-iv)-(7-vi).

(3) Without loss of generality, we assume $b \leq a$. By the proof of Theorem 4.1(3), we can choose $0 < c_1 < x_0 < c_2$ and $L > 0, K > 0$, such that S defined as (4.3) has a fixed point u_1^* satisfying $c_1 \leq u_1^*(t) \leq x_0$ and Q defined as (4.5) has a fixed point u_2^* satisfying $x_0 \leq u_2^*(t) \leq c_2$. Clearly, $u_1^*(t)$ and $u_2^*(t)$ are two positive solutions of (1.1).

(6) By $a > 0, b > 0$, and $\lambda_* < 0$, there exist x_1 and x_2 satisfying $0 < x_1 < x_0 < x_2$ such that $g(x_1) = g(x_2) = 0$. Therefore, if $h(t) \equiv 0$ on $[1, T]_{\mathbb{Z}}$, then $u(t) \equiv x_1$ and $u(t) \equiv x_2$ are two positive solutions of (1.1). If $0 < h(t) < \min\{a, b\}$, we know that there exist $0 < c_1 < c_2 < x_1$ and $x_2 < c_3 < c_4$ such that

$$g(c_1) > h(t), \quad g(c_2) < h(t), \quad g(c_3) < h(t), \quad g(c_4) > h(t).$$

Let $u_i(t) \equiv c_i$ ($i = 1, 2, 3, 4$). Similar to the proof of Theorem 4.1(3), consider the operator S defined as (4.3) on $[u_1, u_2]$ and the operator Q defined as (4.5) on $[u_3, u_4]$, one can find that S has a fixed point in $[u_1, u_2]$ and Q has a fixed point in $[u_3, u_4]$.

(7-iv) By $a \leq 0, b > 0$, then there exists $x_1 > x_0$ by $\lambda_* < 0$ such that $g(x_1) = 0$. It is easy to see that $u(t) \equiv x_1$ is a positive solution of (1.1) since $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

(7-v) By $a \leq 0, b > 0$, there exists $x_1 > 0$ such that $g(x_1) = 0$. Thus, $0 < h(t) < b, t \in [1, T]_{\mathbb{Z}}$, implies that there exist $c_2 > c_1 > x_1$ such that $g(c_1) < h(t)$ and $g(c_2) > h(t)$ for $t \in [1, T]_{\mathbb{Z}}$. Similar to the proof of Theorem 4.1(3), Q defined on $[u_1, u_2]$ has a fixed point in $[u_1, u_2]$, where $u_1(t) \equiv c_1, u_2(t) \equiv c_2, t \in [1, T]_{\mathbb{Z}}$.

(7-vi) By $a > 0, b < 0$, there exists $x_1 > 0$ such that $g(x_1) = 0$. Since $0 < h(t) < a$ for $t \in [1, T]_{\mathbb{Z}}$, there exist $0 < c_1 < c_2 < x_1$ such that $g(c_1) > h(t)$ and $g(c_2) < h(t)$ for $t \in [1, T]_{\mathbb{Z}}$. Similar to the proof of Theorem 4.1(3), S defined on $[u_1, u_2]$ has a fixed point in $[u_1, u_2]$, where $u_1(t) \equiv c_1, u_2(t) \equiv c_2, t \in [1, T]_{\mathbb{Z}}$.

The proof is complete. \square

Finally, if $g \in C^1((0, +\infty), \mathbb{R})$ and g has a unique global maximum value on $(0, +\infty)$, we have the following result.

Theorem 4.4 Let $g \in C^1((0, +\infty), \mathbb{R})$, $\lim_{x \rightarrow 0^+} g(x) = a$, $\lim_{x \rightarrow +\infty} g(x) = b$. Here, a, b may be $-\infty$. Assume that there exists $x_0 > 0$ such that

$$\lambda^* := g(x_0) = \max_{x \in (0, +\infty)} g(x)$$

and $g(x) < \lambda^*$ for all $x \in (0, +\infty) \setminus \{x_0\}$, $\lambda^* > 0$, and $\lambda^* > \min\{a, b\}$.

(1) If one of the following conditions is satisfied, then problem (1.1) has no positive solution:

(1-i) $h(t) \geq \lambda^*$ and $h(t) \neq \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$;

(1-ii) $\min\{a, b\} \geq 0, h(t) \leq \min\{a, b\}$ for $t \in [1, T]_{\mathbb{Z}}$, and $g(x) > \min\{a, b\}$ for all $x \in (0, +\infty)$.

(2) If $h(t) \equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one positive solution.

(3) If one of the following conditions is satisfied, then problem (1.1) has at least two positive solutions:

(3-i) $\max\{a, b, 0\} < h(t) \leq \lambda^*, h(t) \neq \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $(0, x_0)$;

(3-ii) $a < 0, b < 0$, and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

(4) If one of the following conditions is satisfied, then problem (1.1) has at least one positive solution:

- (4-i) $\max\{a, b\} > 0$, $a \neq b$, and $h(t) \equiv \max\{a, b\}$ for $t \in [1, T]_{\mathbb{Z}}$;
- (4-ii) $a > 0$, $b < 0$, and $0 < h(t) < a$ for $t \in [1, T]_{\mathbb{Z}}$;
- (4-iii) $a < 0$, $b > 0$, $0 < h(t) < b$ for $t \in [1, T]_{\mathbb{Z}}$, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ on $(0, x_0)$;
- (4-iv) either $a \geq 0$, $b < 0$ or $a < 0$, $b \geq 0$, and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

5 Exact numbers of solutions and positive solutions

In this section, we establish the results of exact multiplicities of solutions and positive solutions for problem (1.1). First, we consider the exact number of solutions of (1.1).

Note that $4 \sin^2 \frac{\pi}{4T} < \frac{4}{T^2-1}$ for $T > 3$.

If $g \in C^1(\mathbb{R})$ and g is strictly convex on \mathbb{R} , we have the following result by Theorem 3.5 and Theorem 4.1.

Theorem 5.1 Assume $g \in C^1(\mathbb{R})$, $g'(x)$ is strictly increasing and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in \mathbb{R}$. Let

$$\lim_{x \rightarrow \pm\infty} g(x) = +\infty, \quad \lambda_* = \min_{x \in \mathbb{R}} g(x).$$

- (1) If $h(t) \leq \lambda_*$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no solution.
- (2) If $h(t) \equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one solution.
- (3) If $h(t) \geq \lambda_*$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly two solutions.

If $g \in C^1(\mathbb{R})$ and g is strictly concave on \mathbb{R} , then, by Theorem 3.6 and Theorem 4.2, we have the following result.

Theorem 5.2 Assume $g \in C^1(\mathbb{R})$, $g'(x)$ is strictly decreasing, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in \mathbb{R}$. Let

$$\lim_{x \rightarrow \pm\infty} g(x) = -\infty, \quad \lambda^* = \max_{x \in \mathbb{R}} g(x).$$

- (1) If $h(t) \geq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no solution.
- (2) If $h(t) \equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one solution.
- (3) If $h(t) \leq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly two solutions.

Now, we consider the exact number of positive solutions of problem (1.1). First, we consider the case that $g \in C^1((0, +\infty), \mathbb{R})$ and g is strictly convex on $(0, +\infty)$. If $g \in C^1((0, \infty), \mathbb{R})$, we have the following result by Theorem 3.5 and Theorem 4.3.

Theorem 5.3 Assume $g \in C^1((0, \infty), \mathbb{R})$, $g'(x)$ is strictly increasing, and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in (0, \infty)$. Let

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow +\infty} g(x) = +\infty, \quad \lambda_* = \min_{x \in (0, +\infty)} g(x).$$

- (1) If $h(t) \leq \lambda_*$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no positive solution.
- (2) If $h(t) \equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one positive solution.

(3) If one of the following conditions is satisfied, then problem (1.1) has exactly two positive solutions:

- (i) $\lambda_* \geq 0$, $h(t) \geq \lambda_*$, and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$;
- (ii) $\lambda_* < 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$;
- (iii) $\lambda_* < 0$ and $h(t) > 0$ for $t \in [1, T]_{\mathbb{Z}}$.

If $g \in C^1([0, \infty), \mathbb{R})$, we have the following result.

Theorem 5.4 Assume $g \in C^1([0, \infty), \mathbb{R})$, $g'(x)$ is strictly increasing and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in [0, \infty)$. Let

$$g(0) = a, \quad \lim_{x \rightarrow +\infty} g(x) = +\infty, \quad \lambda_* = \min_{x \in [0, +\infty)} g(x).$$

(I) Suppose $\lambda_* \geq 0$.

- (1) If $h(t) \leq \lambda_*$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no positive solution.
- (2) If one of the following conditions is satisfied, then problem (1.1) has exactly one positive solution:
 - (2-i) $\lambda_* < a$ and $h(t) \equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (2-ii) $\lambda_* < a$ and $h(t) \equiv a$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (2-iii) $h(t) > a$ for $t \in [1, T]_{\mathbb{Z}}$.
- (3) If $\lambda_* \leq h(t) < a$ and $h(t) \not\equiv \lambda_*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly two positive solutions.

(II) Suppose $\lambda_* < 0$.

- (4) If one of the following conditions is satisfied, then problem (1.1) has exactly one positive solution:
 - (4-i) $a > 0$ and $h(t) \equiv a$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (4-ii) $a > 0$ and $h(t) > a$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (4-iii) $a \leq 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (4-iv) $a \leq 0$ and $h(t) > 0$ for $t \in [1, T]_{\mathbb{Z}}$.
- (5) If one of the following conditions is satisfied, then problem (1.1) has exactly two positive solutions:
 - (5-i) $a > 0$ and $0 < h(t) < a$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (5-ii) $a > 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

Proof Let $\lambda_* = g(x_0)$. We only show the proofs of (2-ii) and (2-iii).

(2-ii) By g' is strictly increasing, there exists a unique $x_1 > x_0$ such that $g(x_1) = a$ and $g(x) > a$ for all $x > x_1$. Clearly, $u(t) \equiv x_1$ is a positive solution of (1.1). Assume that $v(t)$ is another positive solution, then u and v are strictly ordered by Lemma 3.4. If $x_1 \equiv u(t) < v(t)$, $t \in [1, T]_{\mathbb{Z}}$, then $g(u(t)) \equiv a < g(v(t))$, $t \in [1, T]_{\mathbb{Z}}$, by the fact that $g(x)$ is strictly increasing on $[x_1, +\infty)$. Thus, $\Delta^2 v(t-1) < 0$, $t \in [1, T]_{\mathbb{Z}}$, which implies $\Delta v(T) < \Delta v(0)$, a contradiction. If $v(t) < u(t) \equiv x_1$, then the fact that $g(x) < a$ for $x \in (0, x_1)$ shows that $g(v(t)) < g(u(t)) \equiv a$ and hence $\Delta^2 v(t-1) > 0$, $t \in [1, T]_{\mathbb{Z}}$. It follows that $\Delta v(T) > \Delta v(0)$, a contradiction.

(2-iii) We distinguish two cases to finish the proof.

Case 1: $\lambda_* = a = g(0)$. In this case, g is strictly increasing on $[0, +\infty)$. It is easy to see that (1.1) has at least one positive solution by a discussion similar to Theorem 4.1(4-iii). Now, we assume that (1.1) has two distinct positive solutions u_1 and u_2 . Then u_1 and u_2

are strictly ordered by Lemma 3.4. Without loss of generality, we assume that $u_1(t) < u_2(t)$, $t \in [0, T+1]_{\mathbb{Z}}$. Then $g(u_1(t)) < g(u_2(t))$, $t \in [0, T+1]_{\mathbb{Z}}$, which implies that $\Delta^2(u_1 - u_2)(t-1) = g(u_2(t)) - g(u_1(t)) > 0$, $t \in [1, T]_{\mathbb{Z}}$. Thus, we have the contradiction that

$$\Delta u_1(T) - \Delta u_2(T) > \Delta u_1(0) - \Delta u_2(0).$$

Case 2: $\lambda_* < a$. By g' is strictly increasing, there exists a unique $x_1 > x_0$ such that $g(x_1) = a$ and $g(x) > a$ for all $x > x_1$. Since $h(t) > a$, there exist c_1 and c_2 : $x_1 < c_1 < c_2$ such that $g(c_1) < h(t)$, $g(c_2) > h(t)$, $t \in [1, T]_{\mathbb{Z}}$. By the proof of Theorem 4.1(4-iii), we know that (1.1) has a positive solution u_1 with $c_1 \leq u_1(t) \leq c_2$, $t \in [1, T]_{\mathbb{Z}}$. Now, we assume that u_2 is also a positive solution of (1.1). Then u_1 and u_2 are strictly ordered by Lemma 3.4.

If $u_2(t) < u_1(t)$, $t \in [1, T]_{\mathbb{Z}}$, then $g(u_2(t)) < g(u_1(t))$, $t \in [1, T]_{\mathbb{Z}}$. In fact, For any given t , if $x_0 < u_2(t) < u_1(t)$, then $g(u_2(t)) < g(u_1(t))$ since g is strictly increasing on $[x_0, +\infty)$. If $0 < u_2(t) \leq x_0$, then we also have $g(u_2(t)) \leq a = g(x_1) < g(c_1) \leq g(u_1(t))$, $t \in [1, T]_{\mathbb{Z}}$. Thus, $\Delta^2(u_1 - u_2)(t-1) = g(u_2(t)) - g(u_1(t)) < 0$, $t \in [1, T]_{\mathbb{Z}}$. It follows that $\Delta u_1(T) - \Delta u_2(T) < \Delta u_1(0) - \Delta u_2(0)$, a contradiction.

If $u_2(t) > u_1(t)$, $t \in [1, T]_{\mathbb{Z}}$, then, by the monotony of g on $[x_0, +\infty)$, $g(u_1(t)) < g(u_2(t))$, $t \in [1, T]_{\mathbb{Z}}$, and consequently, $\Delta^2(u_1 - u_2)(t-1) = g(u_2(t)) - g(u_1(t)) > 0$, $t \in [1, T]_{\mathbb{Z}}$, which also yields the contradiction that $\Delta u_1(T) - \Delta u_2(T) > \Delta u_1(0) - \Delta u_2(0)$.

The proof is complete. \square

Finally, we consider the case that $g \in C^1((0, +\infty), \mathbf{R})$ and g is strictly concave on $(0, +\infty)$. If $g \in C^1((0, +\infty), \mathbf{R})$, we have the following result by Theorem 3.6 and Theorem 4.4.

Theorem 5.5 Assume $g \in C^1((0, \infty), \mathbf{R})$, $g'(x)$ is strictly decreasing and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in (0, \infty)$. Let

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow +\infty} g(x) = -\infty, \quad \lambda^* = \max_{x \in (0, +\infty)} g(x) \geq 0.$$

- (1) If $h(t) \geq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no positive solution.
- (2) If $h(t) \equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has exactly one positive solution.
- (3) If one of the following conditions is satisfied, then problem (1.1) has exactly two positive solutions:
 - (3-i) $0 < h(t) \leq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$;
 - (3-ii) $\lambda^* > 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

If $g \in C^1([0, +\infty), \mathbf{R})$, we have the following result.

Theorem 5.6 Assume $g \in C^1([0, \infty), \mathbf{R})$, $g'(x)$ is strictly decreasing and $g'(x) < 4 \sin^2 \frac{\pi}{4T}$ for all $x \in [0, \infty)$. Let

$$g(0) = a, \quad \lim_{x \rightarrow +\infty} g(x) = -\infty, \quad \lambda^* = \max_{x \in [0, +\infty)} g(x) > \max\{0, a\}.$$

- (1) If $h(t) \geq \lambda^*$ and $h(t) \not\equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$, then problem (1.1) has no positive solution.
- (2) If one of the following conditions is satisfied, then problem (1.1) has exactly one positive solution:

- (2-i) $h(t) \equiv \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$;
 (2-ii) $a > 0$ and $h(t) \equiv a$ for $t \in [1, T]_{\mathbb{Z}}$;
 (2-iii) $a > 0$ and $0 < h(t) < a$ for $t \in [1, T]_{\mathbb{Z}}$;
 (2-iv) $a \geq 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.
 (3) If one of the following conditions is satisfied, then problem (1.1) has exactly two positive solutions:
 (3-i) $\max\{0, a\} < h(t) \leq \lambda^*$ and $h(t) \neq \lambda^*$ for $t \in [1, T]_{\mathbb{Z}}$;
 (3-ii) $a < 0$ and $h(t) \equiv 0$ for $t \in [1, T]_{\mathbb{Z}}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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